New Design Criteria for Predicting Buckling of Cylindrical Shells under Axial Compression

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New design criteria for predicting buckling loads of thin-walled cylinders subjected to axial compression are presented based on results obtained for shells of elliptical and circular cross section. The criteria have evolved from a combined analytical and experimental program dealing with the effects of various distributions of both asymmetric and axisymmetric imperfections in shape on the buckling behavior of cylinders. Included in the investigation was a study of the buckling strength of ring-stiffened cylinders. In general, it is demonstrated that accurate buckling loads can be predicted based on either the power spectral density or root mean square deviation of the imperfection profile.

Nomenclature

A,B = Fourier coefficients, $\frac{2}{L}\int_{-L/2}^{L/2} W(X)\cos\omega_j X dX$ and $\frac{2}{L}\int_{-L/2}^{L/2} W(X)\sin\omega_j X dX$, respectively
A_r = cross-sectional area of ring-stiffener
c = $[3(l - \nu^2)]^{1/2}$
d = ring spacing
E = modulus of elasticity
K,K_1 = p/q_o
K_2 = n/q_o
L = lq_o/R
ℓ = shell length
l_x = $\ell/m = \pi R/2p$, axial half-wave length
l_{xcl} = $\pi R/q_o$
l_{xj} = imperfection half-wavelength corresponding to the jth Fourier term
m,n = number of half-waves and waves in the axial and circumferential directions respectively
N = number of ring stiffeners
P_r = buckling load of ring-stiffened shell
P_{cl} = $2\pi Et^2/c$, classical buckling load of perfect circular cylinder
p = $m\pi R/2l$
q_o = $(R/t)^{1/2}[12(1 - \nu^2)]^{1/4}$, the classical axisymmetric buckling mode wave number
R = shell radius measured to the median surface
R_B = radius of curvature of elliptical cylinder at the ends of the minor axis
$S(\omega)$ = power spectral density
t = shell wall thickness
$\bar{t}$ = $t + NA_r/\ell$
W = w/t
w = radial deviation of the median surface
X,Y = xq_o/R and yq_o/R, respectively
x,y = axial and circumferential coordinates, respectively
ζ = root mean square deviation of the median surface/average shell wall thickness
λ = theoretical buckling stress of imperfect shell/classical perfect shell buckling stress
λ_r = ring-stiffened cylinder buckling load/classical buckling load of equivalent weight perfect cylinder
μ = median surface imperfection amplitude/average shell wall thickness
ν = Poisson's ratio
τ = $n(t/Rc)^{1/2}$
ω_j = l_{xcl}/l_{xj}, nondimensional spatial frequency

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Introduction

ONE of the primary design considerations for cylindrical shells under compressive loading is stability. Current design manuals[1,2] require that classical buckling loads be reduced by an empirical correlation factor which is based on an accumulation of circular cylinder test data through which a conservative curve was fitted. These arbitrary knockdown factors are applied to cylinders of any wall construction due to the lack of better criteria. Since the load reductions are quite large for $R/t \geq 100$, it is of importance to determine if reliable buckling load predictions can be established using other criteria in view of the fact that the considerable overdesign of shells to date accounts for a significant proportion of the structural weight of rockets and aircraft. If it can be demonstrated that analytical methods are now available for accurately predicting buckling loads consistent with experimental observations, it then becomes possible to consider various means of increasing the critical loads by improved design, manufacturing and handling techniques.

Generally, it is recognized that the dominant factor in reducing the buckling load of thin-walled cylinders (of oval[3,4,5] and circular[6,7] cross section) significantly below the classical value is the presence of geometric shape imperfections in the shell wall. Although most practical shell structures have some degree of stiffening, test results in most instances fall well below theory,[8] thus suggesting that imperfections play a major role in governing the buckling behavior of stiffened cylinders. A recent paper[8] has atetmpted to make use of

Koiter's[9] axisymmetric imperfection theory in conjunction with empirical load reduction factors associated with given R/t values. Based on a collection of test data, it was demonstrated that conservative collapse loads can be estimated that are somewhat higher than those associated with previous design calculations. However, the main deficiency in the empirical method is its inflexibility with respect to improved fabrication and handling procedures. On the other hand, because actual shell structures contain a random distribution of imperfections, it is difficult to formulate a rational design approach in the absence of a random imperfection model theory. Consequently, interim design criteria are proposed based on available random axisymmetric imperfection theory[10] and experiment,[7] taking into account the effect of asymmetry due to general shape imperfection distributions. Substantial theoretical and experimental evidence has been accumulated indicating that a critical axial imperfection wavelength exists leading to a minimum buckling load for a given root mean square (rms) value of imperfection amplitude. Furthermore, this critical wavelength is approximately equal to the classical axisymmetric buckling mode for a perfect circular cylinder for arbitrary values of the imperfection amplitude. The present work contains a brief summary of axisymmetric imperfection theory relevant to the formulation of a new design approach, followed by an analysis of the effect of asymmetric imperfections on the buckling behavior of cylinders. Although the analysis is restricted to circular cylinders, it is contended that an oval shell can be considered in terms of an equivalent circular cylinder having a radius corresponding to the minimum curvature of the oval cylinder.

If shape imperfections control buckling of both unstiffened and stiffened cylinders, it is of interest to examine the load reductions associated with stiffened shells relative to equivalent weight unstiffened shells. It was decided to investigate the effect of ring stiffeners on the buckling behavior of cylinders, although the addition of rings to cylindrical shells is primarily directed towards increasing their buckling resistance to lateral pressure. At present, extensive theoretical and experimental results have been published[11–13] indicating that reasonable agreement can be expected between test data and classical linear theory. To serve as a comparison, several geometrically 'near-perfect' ring-stiffened shells were manufactured and tested under axial compression, the results of which are included in this report.

Theory and Experiment

Axisymmetric Imperfections

A general solution for the effect of a uniform distribution of axisymmetric imperfections of the form

$$W_0 = -\mu \cos(\pi x/l_x) \tag{1}$$

was derived by Koiter[9] and subsequently solved[6] for various values of imperfection amplitudes and wavelengths. The following eigenvalue equation was obtained using a Galerkin procedure to determine an upper bound buckling load solution

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \tag{2}$$

where the coefficients

$$A = A(K,\mu,\tau,c) \tag{3}$$

are listed in Appendix A. It was found that a critical imperfection wavelength corresponding to a minimum buckling load for a given value of imperfection amplitude existed which, to a first approximation, is defined by $K = \frac{1}{2}$. A comparison of the lower bound solutions[6] for minimum λ with the solution of Eq. (2) for $K = \frac{1}{2}$ (Fig. 1) indicates very little difference in the buckling loads. For cylinders of elliptical cross section, the buckling load reduction resulting from an axial imperfection of the form of Eq. (1) can be estimated by Eq. (2), if the cylinder is sufficiently thin, using an equivalent circular cylinder model having a radius of curvature equal to that at the ends of the minor axis. K must then be calculated from the relation,

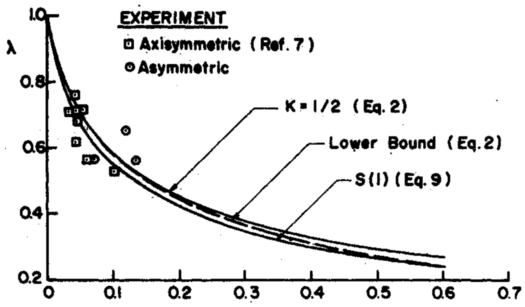

Fig. 1 The effect of random axial imperfection distributions on the buckling of circular cylinders.

$$q_0 = [12(1 - \nu^2)]^{1/4}(R_B/t)^{1/2} \tag{4}$$

A comparison of the predicted response with experiment[4] (Fig. 2) indicates that Koiter's theory can be used for estimating the buckling strength of an axially imperfect elliptical cylinder. It is thus plausible to expect that cylinders of oval cross section can also be analyzed in this manner based on the minimum curvature.

It has recently been shown both theoretically[10] and experimentally[7] that an estimate of the buckling load for a circular cylinder containing a random axisymmetric distribution of imperfections can be obtained from the equation

$$(1 - \lambda)^{7/2} = 9\pi c^2 S(1)\lambda^2/2^{3/2} \tag{5}$$

where $S(1)$ is the power spectral density (PSD) of the imperfection spectrum $S(\omega)$ evaluated at the frequency corresponding to the classical (perfect shell) axisymmetric buckling mode

$$\omega = l_{xcl}/l_x = l \tag{6}$$

When complete generator imperfection profiles are available, Fourier analysis can be employed to estimate $S(1)$ from the relation[7]

$$S(1) \simeq (A_{cr}^2 + B_{cr}^2)\ell q_0/8\pi R \tag{7}$$

where A_{cr}, B_{cr} are the Fourier coefficients evaluated at $\omega = 1$. Although reasonable correlation of theory and experiment was obtained[7] using Eqs. (5) and (7), considerable effort was required to determine the primary contribution to $S(1)$ within a narrow frequency bandwidth centered at $\omega = 1$. In general it was concluded that, providing all imperfection components are of the same order as (or less than) the value at $\omega = 1$, the critical frequency component will govern buckling.

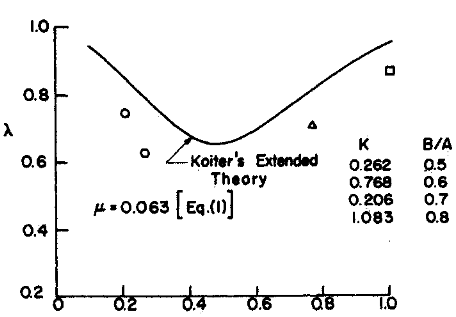

Fig. 2 Comparison of imperfect elliptical cylinder buckling loads with Koiter's extended theory for a range of axial imperfection wavelengths.

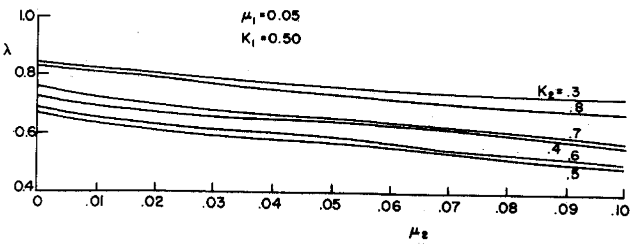

Fig. 3 The effect of asymmetric shape imperfections on the buckling load of a circular cylinder under axial compression.

An alternative formulation for predicting buckling of cylinders with random axisymmetric shape imperfections can be cast in terms of the rms imperfection amplitude by assuming a PSD function. Of particular interest is the spectrum corresponding to an exponential-cosine autocorrelation peaked at the critical frequency ($\omega = 1$). The appropriate form of $S(1)$ is given by[10]

$$S(1) = \zeta^2\beta(1 + \beta^2 + \gamma^2)/\pi[1 + 2(\beta^2 - \gamma^2) + (\beta^2 + \gamma^2)^2] \quad (8)$$

where $\beta = 0.2$ and $\gamma = 1$. Substituting Eq. (8) into Eq. (5) yields

$$(1 - \lambda)^{7/2} = 9c^2\, 2.52\, \zeta^2\lambda^2/2(2)^{1/2} \quad (9)$$

The fact that PSD distributions in actual cylinder configurations will not in general be given by Eq. (8) does not seriously affect the applicability of Eq. (9). This is justified on the basis that the solutions for both the uniform and random axisymmetric imperfection distributions, given by Eq. (2) for $K = \frac{1}{2}$ and Eq. (9) respectively, yield comparable buckling load reductions, as shown in Fig. 1. Since the uniform axisymmetric imperfection corresponding to $K = \frac{1}{2}$ represents the most degrading case (to a first approximation), assuming the measured rms imperfection occurs at $\omega = 1$ should result in a conservative buckling load estimate. It is evident from the results shown in Fig. 1 that theory and experiment are in reasonable agreement.

Because shape imperfections present in practical shell structures are likely to be random in nature, the effects of asymmetric distributions on the buckling behavior must next be examined.

Asymmetric Imperfections

Although some theoretical work[14,15,17] has been done to determine the influence of asymmetric imperfection distributions on the buckling of circular cylinders, general analytical solutions for arbitrary imperfection amplitudes and wavelengths have not been presented. Assuming an initial shape imperfection of the form

$$W_0 = -\mu_1 \cos 2K_1X + \mu_2 \cos K_1X \cos K_2Y \quad (10)$$

buckling load results for a circular cylinder subjected to axial compression have been obtained[16] as a function of μ_1, μ_2, K_1 and K_2 following the basic analysis reported by Hutchinson.[14] Solutions to the nonlinear compatibility and equilibrium equations were obtained by first solving for the Airy stress function in terms of the assumed radial displacement buckling mode

$$W = W_1 \cos 2K_1X + W_2 \cos K_1X \cos K_2Y + W_3 \sin K_1X \cos K_2Y \quad (11)$$

neglecting prebuckling edge constraint effects. The Galerkin procedure was then applied to the equilibrium equation to obtain

$$(8K_1^4 + \tfrac{1}{2} - 4\lambda K_1^2)W_1 + 4\lambda\mu_1K_1^2 + cK_2^2[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{8}](W_2^2 - W_3^2) + cK_2^2[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{4}]W_2\mu_2 = 0 \quad (12)$$

$$\{(K_1^2 + K_2^2)^2/4 + K_1^4/4(K_1^2 + K_2^2)^2 - 2cK_1^4K_2^2\mu_1/(K_1^2 + K_2^2)^2 - \lambda K_1^2/2 + cK_2^2[2K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{4}]W_1\}W_2 + cK_2^2[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{4}]W_1\mu_2 - \lambda\mu_2K_1^2/2 = 0 \quad (13)$$

$$\{(K_1^2 + K_2^2)^2/4 - \lambda K_1^2/2 + 2c\mu_1K_1^4K_2^2/(K_1^2 + K_2^2)^2 - 2cK_2^2W_1[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{8}] + K_1^4/4(K_1^2 + K_2^2)^2\}W_3 = 0 \quad (14)$$

where terms of the order of $W_1^2\mu_1, W_2\mu_2^2$, etc. have been neglected as being small. If the bracketed quantity in Eq. (14) is not equal to zero, then $W_3 = 0$. Hence, substituting for W_1, as determined from Eq. (12), into Eq. (13) and setting $W_3 = 0$ yields

$$D_1 - D_2\lambda + D_3\lambda^2 = 0 \quad (15)$$

or

$$\lambda = [D_2 \pm (D_2^2 - 4D_1D_3)^{1/2}]/2D_3$$

where the coefficients of Eq. (15) (defined in Appendix B) are functions of W_2. Following the procedure outlined by Hutchinson,[14] Eq. (15) was solved for given values of μ_1, μ_2, K_1 and K_2 by increasing W_2 from zero until a maximum λ occurred corresponding to the asymmetric imperfect shell buckling load. A plot of λ vs μ_2 for $\mu_1 = 0.05$, $K_1 = \frac{1}{2}$ and varying K_2 is contained in Fig. 3. It is quite apparent from these results that little variation in the buckling load occurs, over the range of μ_2 considered, relative to the reduction associated with the axial imperfection component (defined by $\mu_2 = 0$). However, it should be noted that significant load reductions have been obtained for large amplitude asymmetric imperfection distributions.[17,18] A critical combination of imperfection wavelengths was found leading to a minimum buckling load, as shown in Fig. 4. This particular set of wavelengths represents a solution of the classical perfect cylinder buckling wave number equation

$$K_1^2 - K_1/2 + K_2^2/4 = 0 \quad (16)$$

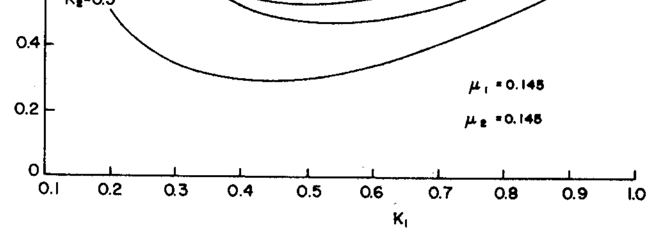

Fig. 4 The effect of asymmetric shape imperfections of varying wave numbers on the buckling load of a circular cylinder under axial compression.

To determine the effect of asymmetry experimentally, circular cylinders were constructed (using the methods described in Refs. 6 and 7) having an initial shape imperfection comprised of a simple sine wave variation circumferentially and a random distribution axially. Median surface profiles for one of the shells are contained in Fig. 5. A comparison of the experimental buckling data with theory [given by Eq. (9) for

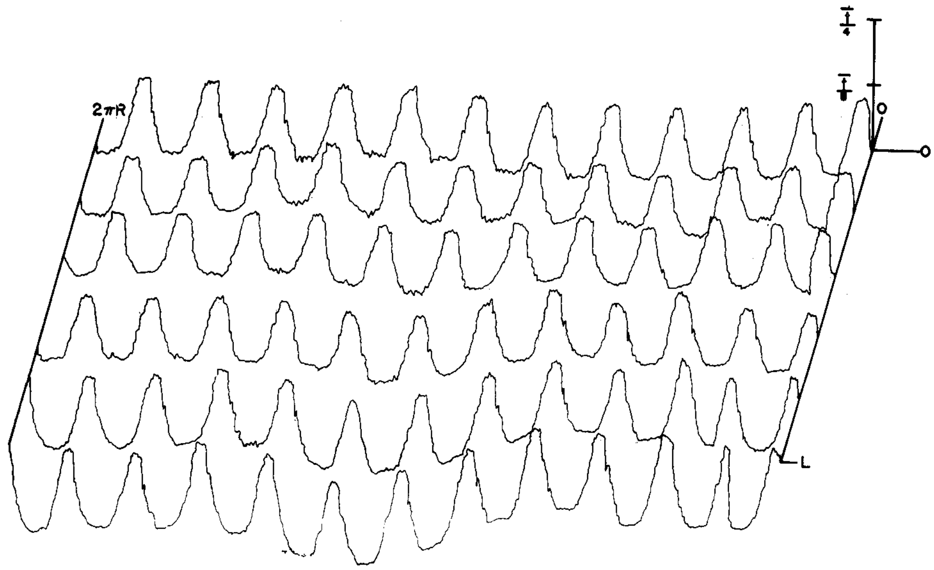

Fig. 5a Circumferential distribution of imperfections.

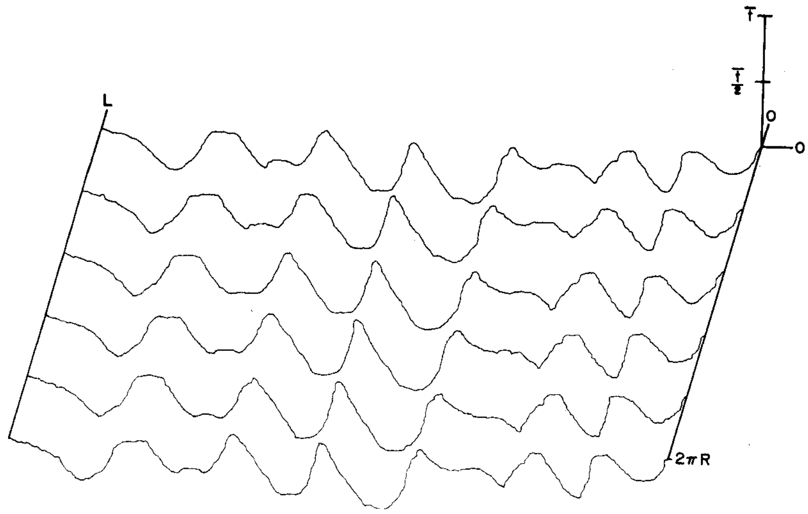

Fig. 5b Random distribution of axial imperfections.

the random axisymmetric case] is shown in Fig. 1 in which the largest value of the measured rms imperfection was used. Although only a few test cases have been studied, it would appear that the effect of asymmetric imperfection distributions can be readily taken into account.

Ring-Stiffened Circular Cylinders

In analyzing the effect of ring stiffeners on the buckling behavior of circular cylindrical shells in axial compression, it is of interest to compare the classical linear buckling load of a ring-stiffened cylinder with the corresponding critical load for an equivalent weight unstiffened shell of thickness,

$$\bar{t} = t + NA_r/\ell \quad (17)$$

Equation (17) was obtained by smearing out the N rings of equal cross-sectional area over the cylinder length. For the case of external ring stiffening, the minimum critical load is associated with an initial buckling mode which is predominantly axisymmetric,[11] although the difference between the axisymmetric and non-axisymmetric solutions is small.[11] Thus, the classical ring-stiffened cylinder buckling load can be estimated from the following equation[11]

$$P_r = P_{cl}(1 + A_r/dt)^{1/2} \quad (18)$$

providing the ring spacing is sufficiently smaller than the axisymmetric buckling mode axial wavelength

$$2\pi(Rt)^{1/2}[12(1 - \nu^2)(1 + A_r/dt)]^{-1/4} \quad (19)$$

to prevent local instability between rings. It is evident from Eq. (18) that the magnitude of the ring eccentricity does not affect the buckling load. A meaningful comparison of classical buckling loads can now be made by dividing Eq. (18) by the unstiffened perfect shell critical load $\bar{P}_{cl}$ (based on $\bar{t}$) to

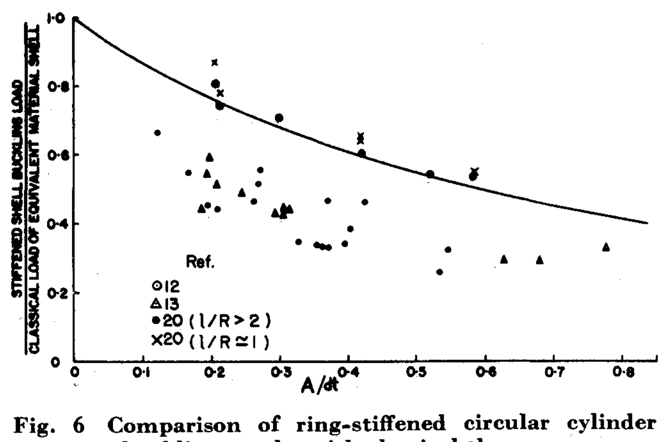

Fig. 6 Comparison of ring-stiffened circular cylinder buckling results with classical theory.

obtain

$$\lambda_r = (t/\bar{t})^2(1 + A_r/dt)^{1/2} \tag{20}$$

If the rings are positioned in such a manner that

$$\ell = Nd \tag{21}$$

then substituting Eqs. (17) and (21) into Eq. (20) yields

$$\lambda_r = (1 + A_r/dt)^{-3/2} \tag{22}$$

Hence, it can be seen that the maximum compressive buckling strength is achieved by an equivalent weight unstiffened cylinder, neglecting the effects of edge constraint and initial shape imperfections. For cylinders containing inside ring stiffeners, it has been shown[11,19] that buckling loads decrease with increasing eccentricity and, in all cases, are less than the value given by Eq. (22) if the shell is sufficiently long. Consequently, Eq. (22) represents a conservative estimate of the reduction in buckling strength due to ring stiffening relative to an equivalent weight cylinder, as confirmed by the experimental results shown in Fig. 6. From the test results[12,13] it is evident that, on the average, there is a 30–40% difference between experiment and theory. Since the prebuckling deformations due to the clamped edge constraints probably account for nominally 10%, the remaining discrepancy must arise from the presence of shape imperfections. Recent tests[20] on geometrically 'near-perfect' ring-stiffened cylinders (Fig. 6) indicate that this is indeed the case. It should be noted that although these particular experimental results lie somewhat above theory, as given by Eq. (22), the ring spacing did not satisfy the condition assumed in Eq. (21). If λ_r is calculated from Eq. (20) based on the cylinder geometry, predicted loads exceeding those given by Eq. (22) are obtained in agreement with experiment.

Proposed Design Criteria

Although insufficient test results have been obtained to date to justify the immediate practical use of the following design procedures, it is proposed that sufficient evidence has been accumulated to warrant extensive testing in conjunction with imperfection distribution measurements in order to realize the full potential of light-weight shell structures.

Unstiffened Cylinders

It is proposed that Eqs. (5) and (9) be used to calculate the critical buckling load of an unstiffened circular cylindrical shell subjected to axial compression by determining the rms or PSD distribution of the shape imperfections. The rms method is by far the simplest procedure to employ in a practical measuring system for large shell structures and requires no interpretation of the significant imperfection components closest to the critical frequency. However, since the *rms* method yields no information on the imperfection components in the spectrum, it cannot be used for refining the manufacturing and handling techniques by reducing the critical frequency terms ($\omega = 1$).

For cylinders of oval cross section, it is proposed that the same equations can be used to predict the collapse strength, noting that the critical frequency can be estimated approximately by Eq. (4), providing the cylinders are sufficiently thin to employ the equivalent circular cylinder model.

Because the effects of asymmetric shape imperfections on reducing the buckling load of a circular cylinder have been shown theoretically to be relatively small compared to the axial imperfection component, the largest value of the PSD or rms as obtained from several (a sufficient number of generator profiles must be analyzed to ensure that the rms or PSD used in the calculation is, within a certain probability, the largest value) generator profile traverses can be used to estimate the buckling load. It is realized that eventually the best design criterion will involve a cross correlation function in order to adequately assess the coupling between the axial and circumferential imperfection components.

Ring-Stiffened Cylinders

Based on the limited test results in Fig. 6, it would appear that ring-stiffened cylinders are imperfection sensitive as noted in Ref. 19. Consequently, the loss in buckling load capability of a ring-stiffened cylinder relative to an equivalent weight unstiffened cylinder in combination with the imperfection sensitivity should be taken into account. From a design point of view, an equivalent weight unstiffened shell having the same wall shape imperfections would buckle at a higher load.

Conclusions

New design criteria have been proposed for predicting the buckling strength of cylindrical shells under axial compression based on the application of power spectral density and root mean square imperfection measurements. The effect of asymmetry due to general shape imperfection distributions can be taken into account by using the largest value of the PSD or rms to yield an accurate estimate of the buckling loads. Recourse to empirical design formulas with arbitrary knock-down factors can eventually be eliminated in favor of the proposed criteria providing one is willing to make the effort required to measure the PSD and rms values associated with a given shell structure.

Appendix A

The coefficients of the eigenvalue equation

$$A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0 \tag{A1}$$

are

$$A_1 = -512K^6Q^2$$

$$A_2 = 64K^4Q^4 + 1024K^8 + 128K^4BQ^2 + 128c\mu\tau^2K^4Q^2$$

$$A_3 = -16K^2BQ^4 - 256K^6B - 8K^2Q^2B^2 - 16c\mu\tau^2K^2Q^2B + 512c\mu\tau^2K^6B$$

$$A_4 = Q^4B^2 + 16K^4B^2 - 64c\mu\tau^2K^4B^2 + 64(c\mu)^2K^4Q^2\tau^4B^2H$$

$$Q = 2K^2 + \tau^2,\ B = 16K^4 + 1$$

$$H = 1/Q^2 + 1/(18K^2 + \tau^2)^2 \tag{A2}$$

Appendix B

The equilibrium equation describing the buckling load solution for a circular cylinder having an asymmetric distribution of geometric shape imperfections is given by

$$D_1 - D_2\lambda + D_3\lambda^2 = 0 \tag{B1}$$

where the coefficients of Eq. (B1) are,

$$D_1 = (B_1A_1 - B_4A_5)W_2 - (B_3A_5 + B_4A_4)W_2^2 - B_3A_4W_2^3$$

$$D_2 = (B_2A_1 + A_2B_1 + B_3A_3)W_2 + B_4A_3 + A_1B_5$$

$$D_3 = A_2B_2W_2 + A_2B_5 \tag{B2}$$

and

$$A_1 = 8K_1^4 + \tfrac{1}{2}$$
$$A_2 = 4K_1^2$$
$$A_3 = -4K_1^2\mu_1$$
$$A_4 = -cK_2^2[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{8}]$$
$$A_5 = -cK_2^2[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{4}]\mu_2$$
$$B_1 = (K_1^2 + K_2^2)^2/4 + K_1^4/4(K_1^2 + K_2^2)^2 - 2cK_1^4K_2^2\mu_1/(K_1^2 + K_2^2)^2$$
$$B_2 = K_1^2/2$$
$$B_3 = -cK_2^2[2K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{4}]$$
$$B_4 = -cK_2^2[K_1^4/(K_1^2 + K_2^2)^2 + \tfrac{1}{4}]\mu_2$$
$$B_5 = K_1^2\mu_2/2 \qquad \text{(B3)}$$

A comparison of the general solution given by Eq. (B1) with the special case solved by Hutchinson[14] can be made by setting $K_1 = K_2 = \frac{1}{2}$ in Eqs. (B3) and subsequently evaluating the coefficients in Eqs. (B2). Equation (B1) then becomes,

$$(W_2 + \mu_2)\lambda^2 - [2W_2 + \mu_2 + c\mu_1(\mu_2 + W_2/2)]\lambda + W_2 - cW_2\mu_1 - (3cW_2/2 + c\mu_2)(3cW_2^2/32 + cW_2\mu_2/8) = 0 \qquad \text{(B4)}$$

Equation (B4) is identical to that obtained by Hutchinson[14] for the case of no internal pressure acting in the cylinder. When $\mu_2 = 0$, the solution of Eq. (B1) does not correspond exactly to the solution of Eq. (2) due to the approximations involved in obtaining Eq. (B1). However, the differences are small for $\mu_1 \leq 0.1$.

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